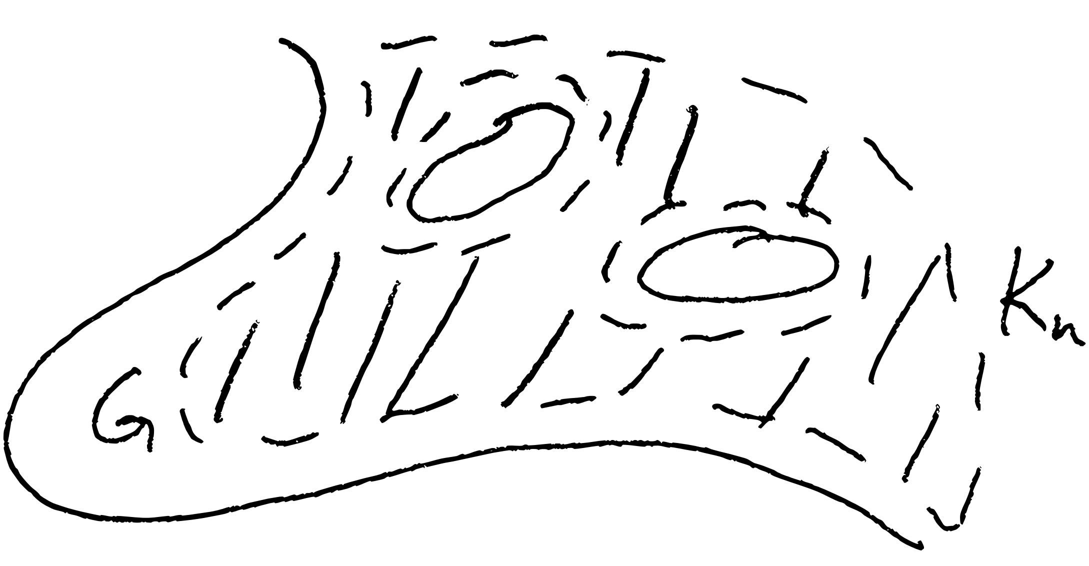
Lecture 7 (1/19/22)

Space of continuous functions

Let (Ω, d) be a metric space and $G \subseteq \mathbb{C}$ an open set. The space of continuous functions $f : G \to \Omega$ is denoted $\mathcal{C}(G, \Omega)$. We shall define a metric ρ on $\mathcal{C}(G,\Omega)$ as follows.

Prop 1. $\exists$ exhaustion of G by compact sets $K_n \subset\subset G$, i.e. $G = \bigcup_{n=1}^{\infty} K_n$, $K_n \subseteq \operatorname{int} K_{n+1}$, and $\forall K \subset\subset G$ $\exists N$ s.t. $K \subseteq K_N$. Moreover, every component of $\mathbb{C}_\infty \setminus K_n$ contains a component of $\mathbb{C}_\infty \setminus G$.



Pf. Set $K_n = \overline{B(0,n)} \cap \{d(z, \mathbb{C}\setminus G) \geq \frac{1}{n}\}$.

Check: • K_n is compact (closed + bdd) ✓

• $\text{int } K_{n+1} = B(0, n+1) \cap \{d(z, \mathbb{C}\setminus G) > \frac{1}{n+1}\}$

$\Rightarrow K_n \subseteq \text{int } K_{n+1}$. ✓

• If $K \subset\subset G$, then $K \subseteq B(0,R)$ and $d(K, \mathbb{C}\setminus G) = \delta > 0$. Thus, if $R \leq n$ and $\frac{1}{n} \leq \delta$ then $K \subset K_n$. ✓

• Same argument as above $\Rightarrow \forall z \; \exists n$ s.t. $z \in K_n \Rightarrow G \subseteq \bigcup_{n=1}^{\infty} K_n$. ✓

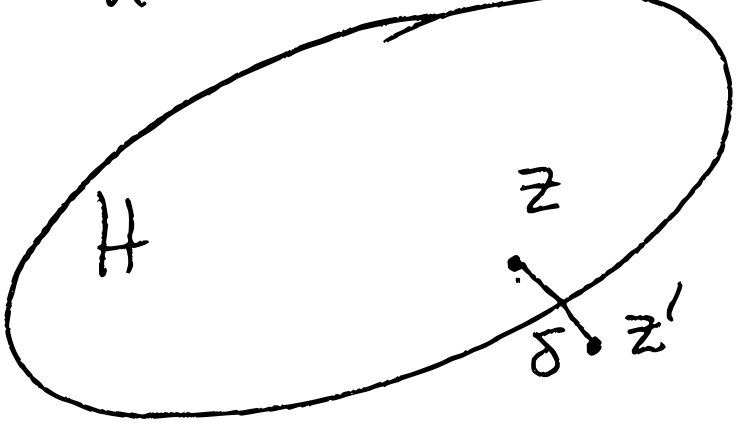
For "Moreover", note that $\mathbb{C}_\infty \setminus K_n = \{|z| > n\} \cup \{d(z, \mathbb{C}\setminus K_n < \frac{1}{n}\} \cup \{\infty\} = B_\infty(\infty, \frac{2}{(1+n^2)^{1/2}}) \cup \{d(z, \mathbb{C}\setminus K_n) < \frac{1}{n}\}$.

↑ Ball at ∞ in FS metric.

The statement for the unbounded component of $\mathbb{C}_\infty \setminus K_n$ is clear since $\infty \in \mathbb{C}_\infty \setminus G$.

Thus, let H be a bdd component.

Since $B_\infty(\infty, \frac{2}{(1+n^2)^{1/2}})$ is contained in the unbdd component of $\mathbb{C}_\infty \setminus K_n$, H is a component of $\{d(z, \mathbb{C}\setminus G) < \frac{1}{n}\}$. Fix $z \in H$. Then $\exists\, z' \in \mathbb{C}\setminus G$ s.t. $d(z, z') = \delta < \frac{1}{n}$.



If $\delta = 0$, then $z = z'$ and $z \in \mathbb{C}\setminus G$ as desired. If $\delta > 0$, consider $B(z', \frac{1}{n})$. <u>Note</u>: $z \in B(z', 1/n)$ and $\forall z'' \in B(z', \frac{1}{n})$:

$$d(z'', \mathbb{C}\setminus G) \leq |z' - z''| < \frac{1}{n} \Rightarrow$$

$z'' \notin K_n$. Since $B(z', \delta')$ connected and H a component $\Rightarrow B(z', \delta') \subseteq H$. Thus, $z' \in H \cap \mathbb{C}\setminus G$ as desired. ☒

__The metric__. Given exhaustion $\{K_n\}$ of G as in Prop 1, for $f, g \in C(G, \Omega)$, let

$$\rho_n(f,g) = \sup_{z \in K_n} d(f(z), g(z))$$

and set

$$\rho(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(f,g)}{1+\rho_n(f,g)} .$$

__Prop 2__ ρ is a metric on $C(G, \Omega)$.

__Pf.__ Check axioms of metric:

- $\rho(f,g) \geq 0$ and "=" iff $f=g$.

  Clearly, $\rho(f,g) \geq 0$ and $\rho(f,f) = 0$.

  If $\rho(f,g) = 0$, then $\rho_n(f,g) = 0$, $\forall n$.

  But then $f=g$ on each K_n. Since $\{K\}_n$ is exhaustion $f=g$ on G. ✓

- $\rho(f,g) = \rho(g,f)$ ✓

- Δ-ineq. Suffices to show Δ-ineq. for each $\dfrac{\rho_n(f,g)}{1+\rho_n(f,g)}$. This is straightforward using Δ-ineq. for each ρ_n and sublinearity of $f(t)=\frac{t}{1+t}$ (HW). ✓

$\boxtimes$

<u>Rmk.</u> Clearly, ρ depends on exhaustion. <u>But</u> as topological space, $\mathcal{C}(G,\Omega)$ does not.

<u>Thm 1.</u> The collection of open sets in $(\mathcal{C}(G,\Omega),\rho)$ does not depend on exhaustion $\{K_n\}_{n=1}^{\infty}$ of G.

We shall in fact prove the following, which readily implies Thm 1.